Stabilization of mechanical systems with underactuation degree one via total energy shaping

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Abstract

Interconnection and damping assignment passivity-based control is a new controller design methodology developed for (asymptotic) stabilization of nonlinear systems that does not rely on, sometimes unnatural and technique–driven, linearization or decoupling procedures but instead endows the closed–loop system with a Hamiltonian structure with a desired energy function—that qualifies as Lyapunov function for the desired equilibrium. The assignable energy functions are characterized by a set of partial differential equations that must be solved to determine the control law. We prove in this paper that for a class of mechanical systems with *underactuation degree one* the partial differential equations can be *explicitly solved*. Furthermore, we introduce a suitable parametrization of assignable energy functions that provides the designer with a handle to address transient performance and robustness issues. Finally, we develop a speed estimator that allows the implementation of *position–feedback* controllers.

Note The present paper is an abridged version of the original work [3] where several examples and all proofs, omitted here, may be found.

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1 Introduction

In [30] we introduced a controller design technique, called interconnection and damping assignment passivity–based control (IDA–PBC), that achieves stabilization for underactuated mechanical systems invoking the physically motivated principles of *energy shaping* and *damping injection*. IDA–PBC endows the closed–loop system with a Hamiltonian structure where the kinetic and potential energy functions have some desirable features, a minimal requirement being to have a minimum at the desired operating point to ensure its stability. Similar techniques have been reported for general port–controlled Hamiltonian and Lagrangian systems in [29, 39] and [31], respectively; see also [12, 13, 14] for the case of Lagrangian mechanical systems and [28] which contains an extensive list of references on this topic. The success of these methods relies on the possibility of solving a set of partial differential equations (PDEs) that identify the energy functions that can be assigned to the closed–loop. The PDE associated to the kinetic energy defines the admissible closed–loop inertia matrices and is nonlinear, while the PDE of assignable potential energy functions is linear. In [12] the authors identify a series of conditions on the system and the assignable inertia matrices such that the PDEs can be solved. Also, techniques to solve the PDEs have been reported in [8, 11] and some geometric aspects of the equations are investigated in [23]. In [18] it is shown that the kinetic energy PDE reduces to an ordinary differential equation (ODE) if the system is of underactuation degree one, that is, if the difference between the number of degrees of freedom and the number of control actions is one—see also [9] for a detailed

study of this case for the Controlled Lagrangian Method. In spite of all these developments the need to solve the PDEs remains the main stumbling block for a wider applicability of these methods.

In this paper we are interested in the application of IDA–PBC to *mechanical systems with underactuation degree one*. The main contributions of the paper are:

1. Identification of a class of underactuation degree one mechanical systems for which the PDEs of IDA–PBC can be *explicitly solved*. Roughly speaking, we assume that the open–loop systems inertia matrix and the force induced by the potential energy (on the unactuated coordinate) are independent of the unactuated coordinate.

2. Derivation of conditions to effectively assign a minimum to the energy function at the desired operating point—providing in this way a complete constructive procedure for stabilization. The conditions are given in terms of single *algebraic inequality* that measures our ability to influence, through the modification of the inertia matrix, the unactuated component of the force induced by potential energy.

3. Development, using the recently introduced method of Immersion and Invariance [6, 22], of a speed estimator that allows the implementation of the proposed controllers *measuring position* only. To the best of our knowledge, this is the first position–feedback solutions reported for these systems—at this level of generality.

4. Last, but not least, the introduction of a suitable parametrization of assignable energy functions—via two *free functions* and a gain matrix—giving the designer the possibility to address transient performance and robustness issues. In spite of their great practical importance these issues are rarely studied in the literature. Indeed, most of the controllers reported for this class of systems rely on the rather unnatural, technique–driven and fragile operations of linearization and decoupling. Other existing schemes give very little freedom to the designer to tune the controller—basically only the selection of saturation and domination functions or the adjustment of high–gain injection or damping.

2 The IDA–PBC method for (simple) mechanical systems

In this section we briefly review the material of [30] that introduces the IDA–PBC approach to regulate the position of underactuated mechanical systems with total energy

$$H(q,p) = \frac{1}{2}p^\top M^{-1}(q)p + V(q) \tag{1}$$

where $q \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ are the generalized position and momenta, respectively, $M = M^\top > 0$ is the inertia matrix, and V is the potential energy. If we assume that the system has no natural damping, then the equations of motion can be written as[1]

$$\begin{bmatrix} \dot q \\ \dot p \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u, \tag{2}$$

where $u \in \mathbb{R}^m$ and $G \in \mathbb{R}^{n\times m}$ with rank$G = m < n$.

In IDA–PBC stabilization is achieved assigning to the closed–loop a desired total energy function. The main result of [30] is contained in the proposition below, that we prove for the sake of completeness.

Proposition 1 *Assume there is $M_d(q) = M_d^\top(q) \in \mathbb{R}^{n\times n}$ and a function $V_d(q)$ that satisfy the PDEs*

$$G^\perp \left\{\nabla_q(p^\top M^{-1}p) - M_d M^{-1}\nabla_q(p^\top M_d^{-1}p) + 2J_2 M_d^{-1}p\right\} = 0 \tag{3}$$

$$G^\perp\{\nabla V - M_d M^{-1}\nabla V_d\} = 0, \tag{4}$$

for some $J_2(q,p) = -J_2^\top(q,p) \in \mathbb{R}^{n\times n}$ and a full rank left annihilator $G^\perp(q) \in \mathbb{R}^{(n-m)\times m}$ of G, i.e., $G^\perp G = 0$ and rank$(G^\perp) = n - m$*. Then, the system (2) in closed–loop with the IDA–PBC*

$$u = (G^\top G)^{-1}G^\top(\nabla_q H - M_d M^{-1}\nabla_q H_d + J_2 M_d^{-1}p) - \\ -K_v G^\top \nabla_p H_d, \tag{5}$$

where $K_v = K_v^\top > 0$, takes the Hamiltonian form

$$\begin{bmatrix} \dot q \\ \dot p \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2 - GK_vG^\top \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}, \tag{6}$$

where the new total energy function is

$$H_d(q,p) = \frac{1}{2}p^\top M_d^{-1}(q)p + V_d(q). \tag{7}$$

Further, if M_d is positive definite in a neighborhood of $q^\star$ and

$$q^\star = \arg\min V_d(q), \tag{8}$$

then $(q^\star, 0)$ is a stable equilibrium point of (6) with Lyapunov function H_d. This equilibrium is asymptotically stable if it is locally detectable from the output $G^\top(q)M_d^{-1}(q)p$. An estimate of the domain of attraction is given by $\Omega_{\bar c}$ where $\Omega_c \triangleq \{(q,p) \in \mathbb{R}^{2n} \mid H_d(q,p) < c\}$ and $\bar c$ corresponds to the largest bounded sub-level set of H_d.

[1] All vectors in the paper are *column* vectors, even the gradient of a scalar function: $\nabla_{(\cdot)} = \frac{\partial}{\partial(\cdot)}$—when clear from the context the subindex in ∇ will be omitted. We will also assume that all functions are sufficiently smooth and, whenever rank conditions are imposed, we assume that they hold uniformly with respect to their arguments.

The main contribution of the present paper is the identification of a class of mechanical systems for which we can explicitly solve the PDEs (3), (4). In spite of the presence of the free matrix J_2, the kinetic energy PDE (3) is a complicated nonlinear matrix PDE. In order to solve it we propose in this paper to *fix* M_d transforming the PDE into an *algebraic equation* that we will solve for J_2. Towards this end, we make first the assumption that the inertia matrix M does not depend on the unactuated coordinates, thus eliminating the term $G^\perp \nabla_q(p^\top M^{-1}p)$ of (3). Second, introducing suitable parameterizations for J_2 and M_d, we will prove that—for the case of underactuation degree one—we have *enough degrees of freedom* in J_2 to solve the algebraic equations. These developments are presented in Section 3.

The potential energy PDE (4), even though linear, may also be difficult to solve analytically. To be able to provide an explicit solution we impose in Section 5 the additional assumption that the unactuated component of the force induced by the potential energy, that is $G^\perp \nabla V$, is a function of *only one* of the actuated coordinates and make M_d a function of this coordinate as well. Stability will be established if we can assign a potential energy function V_d that satisfies (8). See Point 2 in Section 1 and Remark 1 below.

Remark 1 It is clear that, for position regulation problems, our main objective is to shape the potential energy function hence we could leave $M_d = M$ and (4) becomes $G^\perp(\nabla V - \nabla V_d) = 0$. If the systems is underactuated our ability to modify V in this way is obviously limited, see Remark 4.3.18 of [39] and [23]. To overcome this obstacle it was proposed in IDA–PBC [30] to change also the kinetic energy term.[2] This is done through the modification of M—that introduces the "coupling term" $M_d M^{-1}$ in the potential energy PDE. Our objective is then to find, among the set of positive definite M_d that solve (3), one that will allow us to shape V. The key player in this intertwined game is J_2, that we recall is free, thus providing degrees of freedom to assign M_d. See Remark 3 below and [28] for additional discussions on the role of J_2 applications beyond the realm of mechanics.

Remark 2 The class considered in the paper contains several practically relevant examples. A particular case of this class has been studied in [3], and a complete characterization of all underactuation degree one mechanical systems which are feedback–equivalent to it is given in [2].

Remark 3 In the light of some recent misleading novelty claims reported in [40] we find necessary to clarify—again—the history of the term J_2 and its role on stabilization. Already in the first publication concerning IDA–PBC [29] we indicated that, due precisely to the freedom in the choice of this term (that is intrinsic to IDA–PBC), the class of mechanical systems stabilized with IDA–PBC *strictly contains* the class stabilized via the controlled Lagrangian method of [12] or its extension [13]. It was shown that both methods coincide for a particular choice of J_2. This term was given an interpretation in terms of *gyroscopic forces* in a Lagrangian framework for the first time in [11], with a preliminary report widely distributed to the community as early as October 2000. As openly recognized in the Introduction of [16], our work heavily inspired the modified controlled lagrangian method reported in [16], and utilized in [40]—that essentially mimics our derivations.

3 Solving the kinetic energy PDE

We now proceed to define the class of mechanical systems for which we can explicitly solve (3). Toward this end we introduce the following:

Assumption A.1 The system has underactuation degree one, that is, $m = n - 1$.

Assumption A.2 There exists a full rank left annihilator $G^\perp$ of G such that

$$G^\perp \nabla_q(p^\top M^{-1}p) = 0. \tag{9}$$

Assumption A.2 essentially imposes that M does not depend on the unactuated coordinate. It is satisfied by many well–known physical examples, for instance, the Ball and Beam [20], the VTOL Aircraft [25] and the Acrobot [36]. It is easy to see that the assumption is *always satisfied*, taking (with some minor loss of generality) $G = \begin{bmatrix} I_{n-1} \\ 0 \dots 0 \end{bmatrix}$ and introducing a partial feedback–linearization inner–loop [36]. Indeed, after some simple calculations we see that the partially feedback–linearized system takes the so–called Spong's Normal Form [19]:[3]

$$\begin{aligned} \dot q &= p \\ \dot p &= \begin{bmatrix} O \\ -\frac{1}{m_{nn}(q)}\psi_u(q,p) \end{bmatrix} + \begin{bmatrix} I_{n-1} \\ -\frac{1}{m_{nn}(q)} m_u^\top(q) \end{bmatrix} u, \end{aligned} \tag{10}$$

where we have partitioned the inertia matrix and defined the function $\psi_u \in \mathbb{R}$ as

$$M = \begin{bmatrix} \star & m_u \\ m_u^\top & m_{nn} \end{bmatrix}, \quad \psi_u \triangleq e_n^\top(\dot M M^{-1}p + \nabla V)$$

with $m_u \in \mathbb{R}^{n-1}, m_{nn} \in \mathbb{R}$ and e_n the n–th vector of the n–dimensional Euclidean basis. This system is in the form (2) with a "new inertia matrix" equal identity, hence satisfying Assumption A.2.

In the sequel we will impose some assumptions on M, V and G to define a class of mechanical systems for which

[2] To the best of the authors' knowledge the first paper where shaping the *total* energy for stabilization of mechanical systems was proposed is [4], see also Chapter 3 of [31].

[3] Since $\begin{bmatrix} O \\ -\frac{1}{m_{nn}(q)}\psi_u(q,p) \end{bmatrix}$ is not necessarily a gradient vector field a partially linearized system may not be in the form (2).

we can solve the PDEs. These assumptions can be considerably simplified if we proceed from Spong's Normal Form. It is well–known that, in contrast to PBC, feedback–linearization is a fragile operation that requires exact knowledge of the systems parameters and states to ensure the "double integrator" structure. Therefore, we prefer to present the assumptions on the original system (2), stating as remarks their implication for the system in Spong's Normal Form.

3.1 An equivalent representation of the PDE

We find convenient to first express (3) in an alternative equivalent form. For, we introduce a suitable parametrization of the free matrix J_2. It is clear from (3) that J_2 should be *linear* in p. We make now the important observation that, without loss of generality (see Remark 4), J_2 can be parameterized in the form

$$J_2 = \begin{bmatrix} 0 & \tilde{p}^\top\alpha_1 & \tilde{p}^\top\alpha_2 & \dots & \tilde{p}^\top\alpha_{n-1} \\ -\tilde{p}^\top\alpha_1 & 0 & \tilde{p}^\top\alpha_n & \dots & \tilde{p}^\top\alpha_{2n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\tilde{p}^\top\alpha_{n-1} & -\tilde{p}^\top\alpha_{2n-3} & \dots & & 0 \end{bmatrix}$$

where the vector functions $\alpha_i(q) \in \mathbb{R}^n,\ i = 1,\dots,n_o,\ n_o \triangleq \frac{n}{2}(n-1)$, are free parameters and we have defined for notational convenience the (partial) coordinate

$$\tilde{p} \triangleq M_d^{-1}p. \tag{11}$$

Alternatively, we can write

$$J_2 = \sum_{i=1}^{n_o} \tilde{p}^\top\alpha_i W_i, \tag{12}$$

with the $W_i \in \mathbb{R}^{n\times n},\ i = 1,\dots,n_o$, defined as follows. First, we construct n^2 matrices of dimension $n \times n$, that we denote $F^{kl} = \{f_{ij}^{kl}\},\ k,l \in \{1,2,\dots,n\}$, according to the rule

$$f_{ij}^{kl} = \begin{cases} 1 & \text{if} \quad j > i,\ i = k \text{ and } j = l \\ 0 & \text{otherwise.} \end{cases}$$

Notice that only n_o matrices are different from zero. Then, we define $W^{kl} \triangleq F^{kl} - \left(F^{kl}\right)^\top$. Finally, we set (in an obvious way)

$$W_1 = W^{12}, W_2 = W^{13}, \dots, W_n = W^{1n},$$
$$W_{n+1} = W^{23}, \dots, W_{n_o} = W^{(n-1)n}.$$

For instance, for the case $n = 3$, for which also $n_o = 3$, we get

$$W_1 \triangleq \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ W_2 \triangleq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$W_3 \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Using this parameterization some simple calculations establish that the term $G^\perp J_2$ that appears in (3) becomes

$$G^\perp(q)J_2(q,p) = \tilde{p}^\top\mathcal{J}(q)\mathcal{A}^\top(q) \tag{13}$$

where we defined

$$\mathcal{J} \triangleq \left[\alpha_1 \vdots \alpha_2 \vdots \dots \vdots \alpha_{n_o}\right] \in \mathbb{R}^{n\times n_o},$$

which is a *free* matrix, and

$$\mathcal{A} \triangleq \left[W_1\left(G^\perp\right)^\top, W_2\left(G^\perp\right)^\top, \dots, W_{n_o}\left(G^\perp\right)^\top\right] \in \mathbb{R}^{n\times n_o}. \tag{14}$$

Proposition 2 *Under Assumptions A.1, A.2 the kinetic energy PDE (3) becomes*

$$\sum_{i=1}^{n}\gamma_i(q)\frac{dM_d}{dq_i} = -[\mathcal{J}(q)\mathcal{A}^\top(q) + \mathcal{A}(q)\mathcal{J}^\top(q)], \tag{15}$$

where[4]

$$\gamma = \mathit{col}\,(\gamma_1,\dots,\gamma_n) \triangleq M^{-1}M_d(G^\perp)^\top \in \mathbb{R}^n \tag{16}$$

Remark 4 An $n \times n$ skew–symmetric matrix contains *at most* n_o non–zero different terms. Hence, the proposed J_2 contains all skew–symmetric matrices which are linear in $\tilde{p}$, that is, all matrices of the form $\sum_{i=1}^n \Omega_i\tilde{p}_i,\ \Omega_i = -\Omega_i^\top$, and the parametrization is done without loss of generality as claimed above.[5]

3.2 A parametrization of M_d that solves the PDE

In this section we present a parametrization of the desired inertia matrix for which there exists a $\mathcal{J}$ that sets to zero the term in brackets of

$$\tilde{p}^\top[\sum_{i=1}^{n}\gamma_i\frac{dM_d}{dq_i} + 2\mathcal{J}\mathcal{A}^\top]\tilde{p} = 0, \tag{17}$$

that we write here for ease of reference as

$$\sum_{i=1}^{n}\gamma_i\frac{dM_d}{dq_i} = -2\mathcal{A}\mathcal{J}^\top, \tag{18}$$

recalling that γ_i, as defined in (16), are functions of M_d. It is important to underscore that the set of M_d that satisfies (18) is strictly contained in the set that satisfies (15)—which, as stated in Proposition 2, characterizes *all solutions* of (3). We decide to work with this smaller set because, as will be shown below, we can in this way give a simple explicit expression for M_d. Of course, all solutions of (18) are solutions of (3).

[4] Notice that, under Assumption A.1, $G^\perp$ is a *row vector*.

[5] The space of skew–symmetric matrices, usually denoted $so(n)$, can be alternatively defined noting that $so(n)$ is isomorphic to $\mathbb{R}^{n_0}$ via the hat operator $\hat{\cdot} : \mathbb{R}^{n_0} \to so(n)$, and then use the basis $\{\hat{e}_1,\dots,\hat{e}_{n_0}\}$.

As explained in the introduction, we solve (18) as an algebraic equation in the unknown $\mathcal{J}$ for a *given* M_d. Towards this end, we note from (14) and skew–symmetry of the matrices W_i that

$$G^{\perp}\mathcal{A} = 0. \tag{19}$$

The equation above indicates that $\mathcal{A} \in \text{Im } G$ which, in view of (18), suggests to select M_d such that $\frac{dM_d}{dq_i} \in \text{Im } G$ as well. The question on whether there will exists $\mathcal{J}$ to solve (18) will depend on the rank of $\mathcal{A}$ as shown in the following simple linear algebra lemma.

Lemma 1 *Consider a matrix $A \in \mathbb{R}^{n\times n_o}$ with $n_o \geq n$, rank $A = n-1$, and such that $w^\top A = 0$ for some $w \in \mathbb{R}^n$. Then, for all vectors $x \in \mathbb{R}^n$ such that $w^\top x = 0$ there exists a vector $y \in \mathbb{R}^{n_o}$ such that $x = Ay$.*[6]

In order to use Lemma 1 we now establish that $\mathcal{A}$ satisfies the required rank condition.

Lemma 2 *For the matrix $\mathcal{A}$ defined in (14) we have*

$$rank\ \mathcal{A} = n-1.$$

To present the main result of this section—a parametrization of M_d such that (3) can be explicitly solved—we require:

Assumption A.3 The input matrix G is function of a single element of q, say q_r, with r an integer taking values in the set $\{1,\ldots,n\}$.

Obviously, the assumption will be always satisfied if it is possible to (via an input change of coordinates and re–ordering of the variables q) transform the input matrix into $G = \begin{bmatrix} I_{n-1} \\ 0\ldots 0 \end{bmatrix}$. On the other hand, referring to Spong's Normal Form (10), we see that the assumption is satisfied for the partially–linearized system if the column of M corresponding to the unactuated coordinate depends only on q_r.

Proposition 3 *Let Assumptions A.1–A.3 be satisfied. Under these conditions, for all desired (locally) positive definite inertia matrices of the form*

$$M_d(q_r) = \int_{q_r^\star}^{q_r} G(\mu)\Psi(\mu)G^\top(\mu)d\mu + M_d^0 \tag{20}$$

where the matrix function $\Psi = \Psi^\top \in \mathbb{R}^{(n-1)\times(n-1)}$ and the constant matrix $M_d^0 \in \mathbb{R}^{n\times n}$, $M_d^0 = (M_d^0)^\top > 0$, may be arbitrarily *chosen, there exists a matrix J_2 such that the kinetic energy PDE (3) holds in a neighborhood of $q_r^\star$.*

[6]The proof of the lemma for the case $n = 2$, hence $n_0 = 1$, follows from basic plane geometry considerations and is omitted for brevity.

4 Solving the potential energy PDE

The potential energy PDE (4) can be written using (16) as

$$\gamma^\top(q)\nabla V_d = s(q) \tag{21}$$

where, to simplify the notation, we have defined the scalar function

$$s \triangleq G^{\perp}\nabla V. \tag{22}$$

This function, that is uniquely determined by the open–loop system, plays a critical role in the stabilization problem and we propose to take a brief pause to analyze it. First of all, notice that for all admissible equilibria $\bar{q}$, we have

$$s(\bar{q}) = 0. \tag{23}$$

This follows from the dynamic equations for momenta in (2), whose right hand side evaluated for $p = 0$ becomes $-\nabla V + Gu$. Secondly, the vector ∇V contains the forces induced by the potential energy, in particular, $G^{\perp}\nabla V$ are those forces that cannot be (directly) affected by the control. Referring back to the original potential energy PDE (4), we recall that the mechanism to shape the potential energy is through the introduction of the term $M_d M^{-1}$. Since we have imposed that M_d depends on a single coordinate it is reasonable to require that s also depends only on q_r, as will be done below.
Once M_d is fixed, γ as given by (16) is also fixed, and equation (21) is a linear PDE that may be solved using, for instance, the techniques of [10]. See the examples worked out in [30]. Since our interest in this paper is to give a *constructive solution* to the stabilization problem we make two additional assumptions to be able to explicitly solve (21).

Assumption A.4 The vector γ and the function s, defined in (16), (22), respectively, are functions of q_r only, with q_r as in Assumption A.3.

Assumption A.5 $\gamma_r(q_r^\star) \neq 0$.

Under Assumption A.3 and with M_d defined by (20) γ is a function of q_r if M is a function of q_r. Clearly, for systems in Spong's Normal Form, where $M = I_n$, Assumption A.4 will be satisfied if ψ_u does not depend on p. Assumption A.5 is a generic condition that is imposed to ensure that the PDE (21) admits a well–defined solution in a neighborhood of $q_r^\star$. This stems from the fact that the γ_i are functions of q_r and, in view of (23), s vanishes at $q_r^\star$.

We are in position to present our next result whose proof follows from the equivalence of (4) and (21) and some direct computations.

Proposition 4 *Let Assumptions A.1–A.5 be satisfied and M_d be given by (20). Under these conditions,* all solutions

of the potential energy PDE (4) are given by

$$V_d(q) = \int_0^{q_r} \frac{s(\mu)}{\gamma_r(\mu)} d\mu + \Phi(z(q)), \tag{24}$$

with γ, s given in (16), (22), respectively, and $z \in \mathbb{R}^n$,[7] *defined as*

$$z(q) \triangleq q - \int_0^{q_r} \frac{\gamma(\mu)}{\gamma_r(\mu)} d\mu, \tag{25}$$

with Φ an arbitrary *differentiable function.*

Remark 5 Propositions 3 and 4 characterize a set of assignable energy functions of the form (1) in terms of the triplet $\{\Psi, M_d^0, \Phi\}$. The construction proposed for M_d ensures only $M_d(q^\star) > 0$. To enlarge the domain of positivity of M_d—and consequently enlarge the domain of stability—suitable selections of Ψ and M_d^0 must be found. The same comment applies to Assumption A.5 that should be satisfied in some (quantifiable, and hopefully big) neighborhood of $q_r^\star$. We note that the functions $\frac{s}{\gamma_r}$ and $\frac{\gamma_i}{\gamma_r}$ appear explicitly in the control law (5) through the term ∇V_d (implicit in ∇H_d).

5 Main stabilization result

In the previous section we proposed a parametrization of the assignable energy functions in terms of the triplet $\{\Psi, M_d^0, \Phi\}$. Here we will impose some additional constraints on these parameters to ensure asymptotic stability of the closed–loop. As expected, for stability we will require (besides positivity of M_d) assignment of the desired minimum to V_d, i.e., (8). To articulate this condition we note first that the change of coordinates $q \rightsquigarrow z + q_r e_r$ is a *diffeomorphism* that preserves the extrema—hence we analyze the potential energy function in these new coordinates, see [30] for a discussion on this issue. Now, from (24), and the fact that $\Phi(z)$ is arbitrary, it is clear that restrictions will only be imposed on the term $\int \frac{s}{\gamma_r}$. Recalling (23) and Assumption A.5 we note that this function already has an extremum at $q_r^\star$. To ensure that it is a minimum we verify that its second derivative, evaluated at $q_r^\star$, is positive. Some simple calculations show that this condition is equivalent to:

Assumption A.6 $\gamma_r(q_r^\star)\frac{ds}{dq_r}(q_r^\star) > 0$.

The assumption has the following interpretation. First, we recall from (22) that s represents the forces induced by the potential energy function that are unactuated. Second, $q_r^\star$ corresponds to an equilibrium that will, typically, be open–loop unstable therefore the open–loop potential energy function V will have a *maximum* at this point and $\frac{ds}{dq_r}(q_r^\star) < 0$. Finally, from (4) and (16) we see that γ_r is the element of the "coupling term", $G^\perp M^{-1} M_d$, through which we can modify the (unactuated coordinates of the) open–loop potential energy (see Remark 1). In summary, Assumption A.6 reflects our ability to shape, for the purposes of stabilization, the potential energy through modification of the kinetic energy.
Interestingly, we will show in the proposition that the only additional condition imposed for *asymptotic* stability is as follows.

Assumption A.7 $|G^\top M^{-1} e_r(q_r^\star)| \neq 0$.

Furthermore, for the particular case of quadratic Φ, a very simple *explicit* expression for the control law is given.

Proposition 5 *Consider the underactuated mechanical system (2) verifying Assumptions A.1–A.3. Assume there exists matrices Ψ and M_d^0 such that Assumptions A.4–A.6 hold with M_d given by (20). Under these conditions, for all differentiable functions Φ the IDA–PBC (5) ensures that the closed–loop dynamics is a Hamiltonian system of the form (6) with total energy function (7), with V_d defined in (24). Moreover, $(q^\star, 0)$ is a* locally stable *equilibrium with Lyapunov function $H_d(q,p)$ provided the root $q_r = q_r^\star$ of $s(q_r)$ is isolated, the function $z(q)$ satisfies*

$$z(q^\star) = \arg\min \Phi(z), \tag{26}$$

and this minimum is isolated. It will be asymptotically *stable if Assumption A.7 holds.*
Furthermore, if we select

$$\Phi(z(q)) = \frac{1}{2}\left[z(q) - z(q^\star)\right]^\top P \left[z(q) - z(q^\star)\right]$$

with $P = P^\top > 0$, the control law is of the form

$$u = A_1(q)PS(q - q^\star) + \begin{bmatrix} p^\top A_2(q_r)p \\ \vdots \\ p^\top A_n(q_r)p \end{bmatrix} + A_{n+1}(q_r) - \\ -K_v A_{n+2}(q_r)p \tag{27}$$

where $K_v = K_v^\top > 0$ is free, $S \in \mathbb{R}^{(n-1)\times n}$ is obtained removing the r–th row from the n–dimensional identity matrix, for some matrices $A_i,\ i = 1, \ldots, n+2$.

Remark 6 To quantify the domain of attraction, e.g., to obtain an (almost) global version of the asymptotic stability claim, we need to rule out the existence of limit cycles in the whole space (q_r, ν) as well as stable equilibria, different from the desired one. This can be done reinforcing Assumption A.7 as follows.

Assumption A.7' $|G^\top M^{-1} e_r s(q_r)| = 0 \Rightarrow q_r = \bar{q}_r$, i.e., an equilibrium for the generalized coordinates

and imposing the following additional condition:

[7] $z(q)$ is the, so–called, characteristic of the homogeneous part of the PDE [10]. Notice that z is an n–dimensional vector but $z_r = 0$. We have introduced this (awkward) definition for notational compactness.

Assumption A.8 Fix $a > 0$ (possibly $a = +\infty$). For all points $\bar{q}_r \in [q_r^\star - a\ ,\ q_r^\star + a],\ \bar{q}_r \neq q_r^\star$ such that $s(\bar{q}_r) = 0$ we have that

$$\gamma_r(\bar{q}_r)\frac{ds}{dq_r}(\bar{q}_r) < 0.$$

The latter ensures that all other equilibria correspond to *maximum or saddle* points of the desired potential energy function, and are henceforth unstable.

6 Implementation of the controller via position feedback

In this section we prove that, using the recently introduced method of Immersion and Invariance [6, 22], we can design a speed estimator that allows the implementation of the proposed controllers *measuring only position* for the following particular class of systems

$$\begin{aligned} \dot{q} &= M^{-1}(q_r)p \\ \dot{p} &= \eta(q_r) + G(q_r)u, \end{aligned} \tag{28}$$

that clearly satisfies Assumptions A.1–A.4 and contains the examples considered [3]. To ensure stability we will impose the (rather weak) additional assumption that the matrix Ψ (that defines M_d) is *bounded*.

Proposition 6 *Consider the system (28) assuming, without loss of generality, that G is bounded.*[8] *Select* bounded Ψ *and M_d^0 in (20) such that Assumptions A.5 and A.6 hold. Define the* position feedback *controller*

$$u = A_1(q)PS(q - q^\star) + \begin{bmatrix} (\hat{p} + \lambda q)^\top A_2(\hat{p} + \lambda q) \\ \vdots \\ (\hat{p} + \lambda q)^\top A_n(\hat{p} + \lambda q) \end{bmatrix} + \tag{29}$$

$$+ A_{n+1} - K_v A_{n+2}(\hat{p} + \lambda q)$$

where $\lambda > 0$, and $\hat{p}$ is an estimate of $p - \lambda q$ generated via

$$\dot{\hat{p}} = \eta + Gu - \lambda M^{-1}(\hat{p} + \lambda q). \tag{30}$$

Then there exists a neighborhood of the point $(q^\star, 0, -\lambda q^\star)$ such that all trajectories of the closed–loop system starting in this neighborhood are bounded and satisfy

$$\lim_{t\to\infty}(q(t), p(t), \hat{p}(t)) = (q^\star, 0, -\lambda q^\star).$$

Furthermore, if Assumption A.7 holds and the full state feedback controller (27) ensures global asymptotic stability then the neighborhood is the whole space $\mathbb{R}^{3n}$, thus boundedness and convergence are global.

[8] This assumption is without loss of generality, because we can always redefine the control signal with a scalar normalizing factor without affecting the stabilizability properties.

7 Conclusions and future research

In this paper we have identified a class of underactuated mechanical systems for which the IDA–PBC design methodology gives a complete *constructive* solution to the stabilization problem—without the need to solve any PDE. The main assumptions made on the system are that it has underactuation degree one and that, roughly speaking, the dynamics that are not directly affected by the control, e.g. "in Ker G", can be modified through the action of one actuated coordinate q_r. The underactuation degree Assumption A.1 is needed to ensure there are enough degrees of freedom in the free IDA–PBC parameter J_2 to solve the kinetic energy PDE as an algebraic equation. Assumptions A.2 and A.3 ensure that we can construct the solution choosing $\frac{dM_d}{dq_r} \in \text{Im } G$. Assumptions A.4 and A.5, needed to solve the potential energy PDE, specify the role of q_r. Finally, Assumption A.6 measures our ability to affect the potential energy function through the modification of M_d.

We have also presented a *position feedback* implementation—with provable stability properties—for a subclass of the class considered in the paper. (In [2] a characterization of all mechanical systems that are feedback–equivalent to this subclass is given in terms of solvability of a set of PDEs with algebraic constraints.) This class contains several practically interesting benchmark examples, some of which are studied in [3].

Besides ensuring asymptotic stability the IDA–PBC methodology provides the designer with some degrees of freedom to improve transient performance and robustness. These degrees of freedom are given in terms of parameterized expressions for the assignable energy functions. More precisely, the total energy function can be effectively shaped via the selection of the scaling matrix Ψ, the constant matrix M_d^0 in the inertia matrix (20) and the choice of the function Φ in the potential energy (24). An additional tuning parameter is the damping injection gain K_v that may be any positive definite (possibly state–dependent) matrix.

For simplicity we have chosen in our simulations a quadratic function Φ for the potential energy, but motivated by other considerations, e.g., input constraints or rate saturations, we could have also taken other (logarithmic or saturated) functions. An advantage of a quadratic function is that the control law takes a very nice expression (27), which consists of the sum of three types of terms that are modulated by functions of the distinguished coordinate q_r:

– ("proportional–like") linear terms on the additional coordinate error $S(q - q^\star)$ that contribute to the potential energy shaping;[9]

[9] We have shown with examples the importance of a suitable selection of the relative weights (the matrix P) of the configuration coordinates.

– ("derivative–like") linear terms in p due to the damping injection that enforce asymptotic stability;

– ("gyroscopic–like") quadratic terms in p that come from the interconnection matrix J_2. These terms, which serve to propagate the damping through the well–known mechanism of feedback interconnection of passive and strictly passive systems [28], are essential for the solution of the present problem. See Remark 3.

Current research is under way to extend the present work in the following directions.

- In [24] we worked out two examples, the Acrobot and the Furuta's Pendulum, that do not satisfy Assumptions A.2 nor A.4. The term, $G^{\perp}\nabla_q(p^\top M^{-1}p)$ introduces a quadratic term in M_d in the kinetic energy PDE, but it can still be solved with a suitable choice of J_2. Similarly, even though Assumption A.4 does not hold, we can solve the potential energy PDE with a machinery specifically tailored for these examples. Developing a general theory for a well–identified class of systems containing these examples is currently under investigation.

- In the proof of asymptotic stability in Proposition 5 we have established that in the residual set Ω the characteristic of the potential energy PDE is constant. This seems to be a geometric property of the PDEs that needs to be further clarified. In particular, it would be desirable to use it to simplify the proof and remove the, rather awkward, Assumption A.7. (We point out that this property of $z(q)$ holds for other classes of mechanical systems—for instance, the Ball–and–Beam and the Acrobot systems which do not satisfy Assumptions A.2 nor A.4.)

- To relax Assumptions A.3 and A.4 we need to explore the complete set of solutions for M_d defined by (3), or equivalently (15). In particular, it seems necessary to make M_d function of all coordinates.

- Working out a general theory without Assumption A.1 seems a difficult task. On one hand, we cannot transform the kinetic energy PDE into an algebraic equation. On the other hand, as indicated in [23], some geometric obstacles that hamper our ability to shape V_d may appear in this case.

- Comparison of the class studied here with the one identified, via elegant geometric conditions, in [12]. See also [11]. Also, it would be interesting to explore the connections with the recent work [19], where the authors consider underactuation degree one mechanical systems with a cyclic coordinate.

- The examples presented in the paper are transformed into Spong's Normal Form via partial feedback linearization. It has been argued in this paper that this operation is fragile so it would be interesting to avoid it. This extension is also of interest if a true position feedback controller on the actual system is to be realized. Toward this end, the result of Section 6 should be extended to a broader class of systems.

- The proposed controllers should be tested experimentally and confronted with other existing schemes. The outcome of this research will be reported elsewhere.

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